On Taylor dispersion in oscillatory channel flows

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We revisit Taylor dispersion in oscillatory flows at zero Reynolds number, giving an alternative method of calculating the Taylor dispersivity that is easier to use with computer algebra packages to obtain exact expressions. We consider the effect of out-of-phase oscillatory shear and Poiseuille flow, and show that the resulting Taylor dispersivity is independent of the phase difference. We also determine exact expressions for several examples of oscillatory power-law fluid flows.

Keywords: Taylor dispersion, oscillatory flows, non-Newtonian fluids, power-law fluids, particle sorting, transport in bones and connective tissue, zero Reynolds number

1. Introduction

The standard Taylor-dispersion problem (Taylor (1953)) is to determine the approximate long-time distribution of some passive tracer with density ϕ in a channel or pipe $\Omega = \mathbb{R} \times S$, where S is some connected smooth one- or two-dimensional manifold. The governing equation in PDE form is

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (V\phi - D\nabla\phi) = 0, \tag{1.1}$$

where t is time, V is the velocity field, and D is the diffusivity, usually with zero-flux boundary conditions on the walls. Here we focus on the case where $S = [0, a]$ and the component of the velocity along the channel is a periodic function of time.

The early work on oscillatory flows focused on flow in pipes (Aris (1960)), and much later work on oceanographic applications in unbounded flows (Young et. al. (1982)), but more recently there has been interest in biomechanics, notably Schmidt et. al. (2005), and the references therein, who studied consequences of oscillatory flows on nutrient transport in bone during physical activity and ultrasonic therapy. In these biomechanical applications, the flow is probably better modelled by a combination of shear- and pressure-driven flows, and natural questions are whether these contributions interact, and whether the phase difference between the two components is important.

Also in this study we find an expression for the Taylor dispersivity that is more appropriate for use with computer algebra packages. This extends the work of Jansons and Rogers (1995), by providing a more accessible version of the solution

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in a form that could be easily applied by people not familiar with the theory of stochastic processes, although it uses stochastic calculus in its derivation.

Using computer algebra, we then find the general expression for the Taylor dispersivity at zero Reynolds number for sinusoidal shear and Poiseuille flow with arbitrary phase difference, and show in particular that the Taylor dispersivity is independent of this phase difference. This extends the work of Claes and Van de Broeck (1991). We also find exact expressions for the Taylor dispersivity at zero Reynolds number for several pressure-driven power-law fluid flows.

2. Dispersion in an oscillatory flow

We suppose that the *vertical* diffusion Y is confined to the interval $[0, a]$, which in Itô form is governed by

$$dY_t = \sigma dB_t + v(Y_t)dt + dL_t^0 - dL_t^a, \tag{2.1}$$

where B is a standard Brownian motion (that is, it has diffusivity $\frac{1}{2}$), and L^0 and L^a are local-time terms to keep the particle in the interval. (The local times change only when Y hits a boundary.) The above equation describes, in fluid mechanical terms, a particle with diffusivity $D = \sigma^2/2$ and vertical drift v. For simplicity, we take σ to be constant, as it is in almost all applications, though this condition could easily be relaxed, and we also, for simplicity, assume that v is bounded.

As is usual in Taylor-dispersion problems, we neglect the effect of particle diffusion in the x direction, though it would not be difficult to include this effect if it were considered useful. Thus the *horizontal* displacement is given by

$$X_t \equiv \int_0^t u(s, Y_s)ds, \tag{2.2}$$

where $u(t, y)$ is a periodic function of time, with period $2\pi/\omega$ say, and "$\equiv$" means equal by definition.

From (2.1), we see that the stationary vertical density q is given by

$$q(y)dy \equiv \lim_{t\to\infty} P[Y_t \in dy] = \alpha \exp\left(\int_0^y \frac{2v(\eta)}{\sigma^2} d\eta\right) dy, \tag{2.3}$$

where α is a normalization constant (so $\int_0^a q(y)dy = 1$).

The part of the horizontal displacement due to the cross-channel mean of horizontal advection is trivial, and unimportant in the long-time limit as it is bounded, so we remove this contribution from the problem by replacing $u(t, y)$ with $u'(t, y) \equiv u(t, y) - \int_0^a q(y)u(t, y)dy$.

To find the variance of X_t, apply Itô's formula:

$$df(t, Y_t) = \frac{\partial f}{\partial y}\left(\sigma dB_t + dL_t^0 - dL_t^a\right) + \left(\tfrac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial y^2} + v\frac{\partial f}{\partial y} + \frac{\partial f}{\partial t}\right) dt, \tag{2.4}$$

for some function f to be identified. Now take $\frac{\partial f}{\partial y}(t, 0) = \frac{\partial f}{\partial y}(t, a) = 0$ for all t to remove the local-time terms, and

$$\tfrac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial y^2} + v\frac{\partial f}{\partial y} + \frac{\partial f}{\partial t} = u'(t, y), \tag{2.5}$$

where f is taken to be an oscillatory function of t, with period $2\pi/\omega$. This gives

$$f(t, Y_t) - f(0, Y_0) = \int_0^t \frac{\partial f}{\partial y}(s, Y_s) dB_s + X'_t, \tag{2.6}$$

where

$$X'_t \equiv \int_0^t u'(s, Y_s) ds. \tag{2.7}$$

The integral term can be rewritten, namely

$$\int_0^t \frac{\partial f}{\partial y}(s, Y_s) dB_s = W\left(\int_0^t \left(\frac{\partial f}{\partial y}(s, Y_s)\right)^2 ds\right), \tag{2.8}$$

where W is a standard Brownian motion. Then the ergodic theorem tell us that in the limit $t \to \infty$,

$$\frac{1}{t}\int_0^t \left(\frac{\partial f}{\partial y}(s, Y_s)\right)^2 ds \to \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\int_0^a \left(\frac{\partial f}{\partial y}(t, y)\right)^2 q(y) dy\, dt. \tag{2.9}$$

First observe that

$$\lim_{t\to\infty} E[X_t^2]/t = \lim_{t\to\infty} E[X'_t]/t, \tag{2.10}$$

and second that f is bounded. Thus, (2.6), together with the other observations, implies that the Taylor dispersivity is given by

$$\mathcal{D} \equiv \lim_{t\to\infty} E[X_t^2]/t = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\int_0^a \left(\frac{\partial f}{\partial y}(t, y)\right)^2 q(y) dy\, dt. \tag{2.11}$$

Furthermore, in the limit $t \to \infty$,

$$\frac{X_t}{t^{\frac{1}{2}}} \to N(0, \mathcal{D}) \tag{2.12}$$

in distribution.

3. Without vertical advection

Throughout this section, we assume that the vertical advection $v = 0$. Furthermore, suppose that the horizontal advection has the form

$$u(t, y) = u_1(t, y) + u_2(t, y), \tag{3.1}$$

where u_1 and u_2 are both periodic functions of time with period $2\pi/\omega$, and, for all t and y, that $u'_1(t, y) = u'_1(t, a - y)$, and $u'_2(t, y) = -u'_2(t, a - y)$, where again the dashes denote that the y means have been removed. In this case, since (2.5) is linear, f can be split into f_1 and f_2 forced respectively by u'_1 and u'_2, and inheriting the same symmetries in y. Thus (2.11) implies that

$$\mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2, \tag{3.2}$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ are the Taylor dispersivities respectively for u_1 and u_2, since $q = 1/a$ and

$$\int_0^a \frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial y}dy = 0, \tag{3.3}$$

as the integrand is an odd function of $y - a/2$.

Note that (3.2) does not hold in general for arbitrary v, unless the periods of the two oscillations are different.

4. Special cases

We now consider several example flows in which it is possible to find the Taylor dispersivity exactly. We begin by considering the way in which shear- and pressure-driven flows interact in the case of a Newtonian fluid, and then consider several cases of power-law fluid flows. In these examples, the vertical advection velocity v is zero. The former, in effect, provide solutions to the Hele-Shaw cell model of the corresponding porous material system, namely a microscopic flow driven by shearing the whole porous material, and a pressure-driven flow for a fixed porous material. For biomechanical applications, see Schmidt et. al. (2005).

In general we need to solve (2.5), and the algebra is very messy even in simple cases. If v, the vertical advection velocity, is non-zero, even if it is constant, the corresponding expressions for Taylor dispersivity $\mathcal{D}$ are too complicated to show here, even for simple horizontal flows. However, using symbolic algebra packages, it is possible to find simple exact expressions for $\mathcal{D}$ in some cases.

For simplicity in the rest of this section, we choose units that make $\sigma = 1$, and $a = 1$. Also in these special cases $v = 0$, so (2.5) reduces to

$$\tfrac{1}{2}\frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial t} = u'(t, y), \tag{4.1}$$

with $\frac{\partial f}{\partial y}(t, 0) = \frac{\partial f}{\partial y}(t, 1) = 0$ for all t. Since $v = 0$ and $a = 1$, we find $q = 1$, which slightly simplifies (2.11). With this non-dimensionalization, the time-scale for the cross-channel relaxation to occur is 1, and the time-scale of the oscillations is ω^{-1}. In the following examples, u is proportional to $\cos(\omega t)$. Thus, when doing the computer algebra, we can consider u and f to be the real parts of functions proportional to $\exp(i\omega t)$.

(a) $u(t, y) = \cos(\omega t)y$ *and* $v = 0$

For $u(t, y) = \cos(\omega t)y$, using computer algebra, we find from (4.1) and (2.11)

$$\mathcal{D} = d_1 \equiv \frac{\nu\cos(\nu) + \nu\cosh(\nu) - \sin(\nu) - \sinh(\nu)}{2\nu^5(\cos(\nu) + \cosh(\nu))}, \tag{4.2}$$

where $\nu \equiv \omega^{\frac{1}{2}}$ (see Figure 1). In the limit of small ω,

$$\mathcal{D} \sim \frac{1}{60} - \frac{31\omega^2}{45360} + \frac{5461\omega^4}{194594400} + \cdots . \tag{4.3}$$

In the limit of large ω,

$$\mathcal{D} \sim \frac{1}{2\omega^2} + \cdots . \tag{4.4}$$

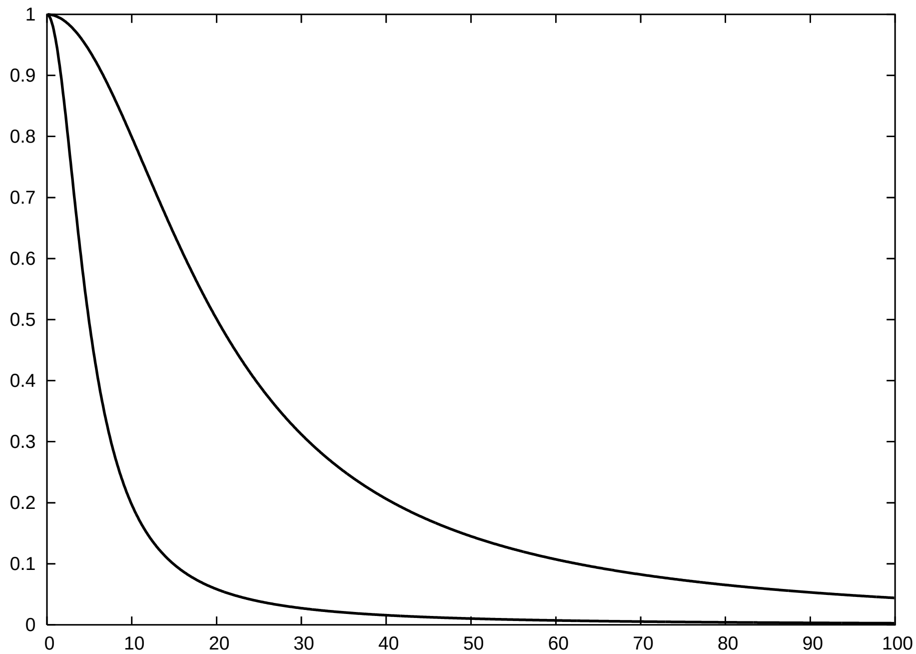

Figure 1. Plots of $\mathcal{D}(\omega)/\mathcal{D}(0)$. The upper curve is for $u(t, y) = \cos(\omega t)y(1-y)$ and the lower curve is for $u(t, y) = \cos(\omega t)y$.

This is in agreement with the result of Claes and Van de Broeck (1991). This is almost, but not exactly, the result previously found by Young et. al. (1982) and extended by Jansons and Rogers (1995) for the case of unbounded flow. It differs by a factor of 2 from the result of Young et. al. (1982), which is explained by Jansons and Rogers (1995), and lacks the oscillatory term of Jansons and Rogers (1995). The large ω result here is, as expected, the average of the result for the unbounded case.

(*b*) $u(t, y) = \cos(\omega t)y(1-y)$ *and* $v = 0$

Similarly for $u(t, y) = \cos(\omega t)y(1-y)$, and solving for $\mathcal{D}$ using computer algebra, we find

$$\mathcal{D} = d_2 \equiv \frac{\nu\cos(\nu) - \nu\cosh(\nu) - 3\sin(\nu) + 3\sinh(\nu)}{6\nu^5(\cos(\nu) - \cosh(\nu))}, \tag{4.5}$$

(see Figure 1). In the limit of small ω,

$$\mathcal{D} \sim \frac{1}{3780} - \frac{\omega^2}{1496880} + \frac{\omega^4}{583783200} + \cdots. \tag{4.6}$$

In the limit of large ω,

$$\mathcal{D} \sim \frac{1}{6\omega^2} + \cdots. \tag{4.7}$$

This agrees with the result obtained by Claes and Van de Broeck (1991).

(c) A linear combination of the flows in the previous examples

We now extend the work of Claes and Van de Broeck (1991), and consider a flow driven by both a moving upper boundary and a pressure gradient, but not necessarily in phase, namely

$$u(t, y) = U_1 \cos(\omega t) y + U_2 \cos(\omega t + \psi) y(1 - y), \tag{4.8}$$

where ψ is a constant. From (3.2), (4.2) and (4.5), we find

$$\mathcal{D} = U_1^2 d_1 + U_2^2 d_2. \tag{4.9}$$

Notice that this result does not depend on the phase shift, which could be important in some applications.

(d) Power-law fluids

The same approach can be used to quickly find exact expressions for some non-Newtonian flows. Here we consider power-law fluid flows, driven by pressure gradients. However, most of the results are too complex to give here, and are essentially useful only in computer programs. However, with the procedure given here, the reader can quickly recalculate them with the help of computer algebra. These flows are of the following form:

$$u_n(t, y) \equiv \cos(\omega t)(\text{const.} - |y - \tfrac{1}{2}|^n), \tag{4.10}$$

where the constant is fixed by the boundary conditions which may be no-slip or partial slip. (Note that n is not the defining parameter of a power-law fluid, but it is more convenient here.) However the results for the Taylor dispersivity does not depend on the constant.

Exact expressions for the Taylor dispersivity can be found for some values of n. However, the expressions for integer n seem to be by far the simplest, and we give a few examples of these below. We were unable to find the solution for general n.

As these flows are symmetrical about $y = \frac{1}{2}$, we solve in a half range which means we avoid the singularity at $y = \frac{1}{2}$, which is problematic for computer algebra.

In the following results, we do not take account of the relaxation times for the fluid, so we must assume that the time-scale of the oscillation is long compared with these.

The case $n = 1$ is a rescaling of the Newtonian shear-flow result above:

$$\mathcal{D} = \frac{\nu \cos(\frac{\nu}{2}) + \nu \cosh(\frac{\nu}{2}) - 2 \sin(\frac{\nu}{2}) - 2 \sinh(\frac{\nu}{2})}{2 \nu^5 \left(\cos(\frac{\nu}{2}) + \cosh(\frac{\nu}{2})\right)}. \tag{4.11}$$

The case $n = 2$ is for a Newtonian fluid, already considered above (see (4.5)):

$$\mathcal{D} = d_2. \tag{4.12}$$

The case $n = 3$:

$$\begin{aligned}
\mathcal{D} = {} & (9\,(\nu\,(-80 + \nu^4)\,\cos(\nu) - \nu\,(-80 + \nu^4)\,\cosh(\nu) \\
& + 80\,(-4 + \nu^2)\,\cosh(\frac{\nu}{2})\,\sin(\frac{\nu}{2}) - 5\,(-32 + 8\,\nu^2 + \nu^4)\,\sin(\nu) \\
& + 80\,(4 + \nu^2)\,\cos(\frac{\nu}{2})\,\sinh(\frac{\nu}{2}) + 5\,(-32 - 8\,\nu^2 + \nu^4)\,\sinh(\nu))) \\
& /(160\,\nu^9\,(\cos(\nu) - \cosh(\nu))).
\end{aligned} \tag{4.13}$$

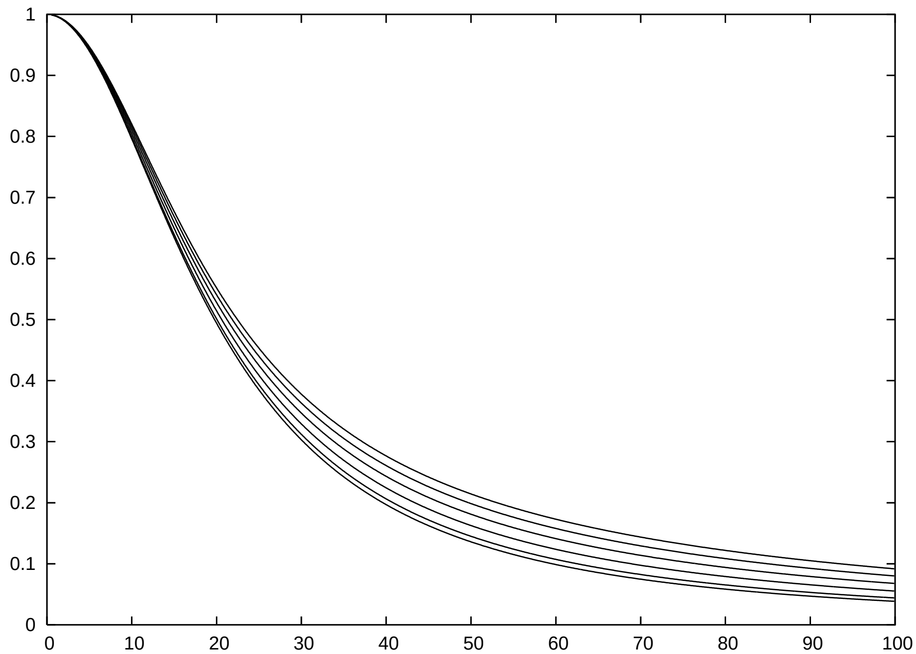

Figure 2. Plots of $\mathcal{D}(\omega)/\mathcal{D}(0)$ for u_n with n increasing from 1 at the bottom to 6 at the top.

The case $n = 4$:

$$\begin{aligned}\mathcal{D} = (&\nu\,(-672+\nu^4)\,\cos(\nu) - \nu\,(-672+\nu^4)\,\cosh(\nu)\\ &- 7\,(-144+24\,\nu^2+\nu^4)\,\sin(\nu) + 7\,(-144-24\,\nu^2+\nu^4)\,\sinh(\nu))\\ &/(56\,\nu^9\,(\cos(\nu)-\cosh(\nu))).\end{aligned} \tag{4.14}$$

The case $n = 5$:

$$\begin{aligned}\mathcal{D} = (&5\,(\nu\,(414720-13824\,\nu^4+5\,\nu^8)\,\cos(\nu)\\ &+ \nu\,(-414720+13824\,\nu^4-5\,\nu^8)\,\cosh(\nu)\\ &- 17280\,(-96+24\,\nu^2+\nu^4)\,\cosh(\frac{\nu}{2})\,\sin(\frac{\nu}{2})\\ &- 45\,(18432-4608\,\nu^2-768\,\nu^4+48\,\nu^6+\nu^8)\,\sin(\nu)\\ &+ 17280\,(-96-24\,\nu^2+\nu^4)\,\cos(\frac{\nu}{2})\,\sinh(\frac{\nu}{2})\\ &+ 45\,(18432+4608\,\nu^2-768\,\nu^4-48\,\nu^6+\nu^8)\,\sinh(\nu)))\\ &/(4608\,\nu^{13}\,(\cos(\nu)-\cosh(\nu))).\end{aligned} \tag{4.15}$$

The case $n = 6$:

$$\begin{aligned}\mathcal{D} = {} & (9\,(\nu\,(11827200 - 54560\,\nu^4 + 7\,\nu^8)\,\cos(\nu) \\ & + \nu\,(-11827200 + 54560\,\nu^4 - 7\,\nu^8)\,\cosh(\nu) \\ & - 77\,(230400 - 38400\,\nu^2 - 2560\,\nu^4 + 80\,\nu^6 + \nu^8)\,\sin(\nu) \\ & + 77\,(230400 + 38400\,\nu^2 - 2560\,\nu^4 - 80\,\nu^6 + \nu^8)\,\sinh(\nu))) \\ & /(39424\,\nu^{13}\,(\cos(\nu) - \cosh(\nu))).\end{aligned} \tag{4.16}$$

Notice that in each case $\omega^2\mathcal{D}$ tends to a constant as $\omega \to \infty$.

There is little point in going beyond $n = 6$ as it is almost plug flow, with almost all the dispersion occurring near the boundaries.

The above results suggest that in the case of integer n there is probably some not too complex generating function for the Taylor dispersivities, but MATHEMATICA was unable to work through the necessary algebra. However, since it would need computer algebra to unpack the generating function, one is probably better off computing the required Taylor dispersivity directly.

A plot of all the results of this section are given in Figure 2. Notice that even though we have considered only integer n, the curves are sufficiently close that interpolation would be expected to well approximate intermediate values over the interesting part of the ω range. Also, given that the case $n = 2$ is known, the fact that the other plots are close to this, and occur in the expected order, is a good consistency check on the other results.

(*e*) *Comment on large and small ω limits*

From (4.1) we can see that, for small ω, the approximate balance is

$$\tfrac{1}{2}\frac{\partial^2 f}{\partial y^2} = u'(t, y), \tag{4.17}$$

and therefore the Taylor dispersivity is approximately the time-average of the non-oscillatory Taylor dispersivities for the instantaneous flows. In the sinusoidal case this is just $\frac{1}{2}$ of the non-oscillatory Taylor dispersivity for the peak u.

For large ω, the approximate balance is

$$\frac{\partial f}{\partial t} = u'(t, y), \tag{4.18}$$

giving an asymptotic Taylor dispersivity

$$\mathcal{D} \sim \frac{1}{2\omega^2}\int_0^1 \left(\frac{dU}{dy}(y)\right)^2 dy, \qquad \text{as } \omega \to \infty, \tag{4.19}$$

where U is defined by $u(t, y) = \cos(\omega t)U(y)$.

5. Conclusions

We have shown that the alternative method for determining the Taylor dispersivity for an oscillatory flow lends itself to exact evaluation using computer algebra.

Figure 1 shows that for the two simple flows considered, namely an oscillatory simple shear flow and an oscillatory Poiseuille flow, both have a strong frequency-dependent Taylor dispersivity. These systems therefore could be used for measuring particle diffusivities more accurately by choosing a frequency that moves the corresponding Taylor dispersivities into the sensitive range. Alternatively, the same physical system could be used for separating particles of different diffusivities. Experimentally the oscillatory simple shear flow is particularly easy to set up in a laboratory.

One interesting direction for future research is applications to transport in bones, and other connective tissues, subjected to oscillatory stresses (for example due to exercise or during ultra sound therapy). It appears that physical processes of the type discussed here are essential for normal function, and a better understanding could result in more effective treatment. These applications however involve much more complex geometries, although would be expected to be qualitatively similar. The solutions here are in effect those for the Hele-Shaw cell model of a porous material. Given we have shown that the Taylor dispersivities for shear- and pressure-driven flows do not interact in a channel flow, it is a natural question to ask if this carries over (at least approximately) to these biomechanical applications, where both effects are likely to be present.

It is clear that many more non-Newtonian examples are possible with this approach. However, it is less clear if there are any nice exact expressions for flows with a non-zero vertical (i.e. cross-channel) component. The intermediate expressions in these cases are much more complex, with more branch cuts, but there does appear to be hope. This is one area in which computer algebra is currently weak.

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